

Riesz transforms of the Hodge-de Rham Laplacian on Riemannian manifolds

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Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property. Let $\vec{\Delta}$ be the Hodge-de Rham Laplacian acting on 1-differential forms. According to the Bochner formula, $\vec{\Delta} = \nabla^* \nabla + R_+ - R_-$ where R_+ and R_- are respectively the positive and negative part of the Ricci curvature and ∇ is the Levi-Civita connection. We study the boundedness of the Riesz transform $d^*(\vec{\Delta})^{-\frac{1}{2}}$ from $L^p(\Lambda^1 T^* M)$ to $L^p(M)$ and of the Riesz transform $d(\vec{\Delta})^{-\frac{1}{2}}$ from $L^p(\Lambda^1 T^* M)$ to $L^p(\Lambda^2 T^* M)$. We prove that, if the heat kernel on functions $p_t(x, y)$ satisfies a Gaussian upper bound and if the negative part R_- of the Ricci curvature is ϵ -sub-critical for some $\epsilon \in [0, 1)$, then $d^*(\vec{\Delta})^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^* M)$ to $L^p(M)$ and $d(\vec{\Delta})^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^* M)$ to $L^p(\Lambda^2 T^* M)$ for $p \in (p'_0, 2]$ where $p_0 > 2$ depends on ϵ and on a constant appearing in the doubling volume property. A duality argument gives the boundedness of the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p \in [2, p_0]$ where Δ is the non-negative Laplace-Beltrami operator. We also give a condition on R_- to be ϵ -sub-critical under both analytic and geometric assumptions.

1 Introduction and main results

Let (M, g) be a complete non-compact Riemannian manifold of dimension N , where g denotes a Riemannian metric on M ; that is, g is a family of smoothly varying positive definite inner products g_x on the tangent space $T_x M$ for each $x \in M$. Let ρ and μ be the Riemannian distance and measure associated with g respectively. We suppose that M satisfies the doubling volume property, that is, there exists constants $C, D > 0$ such that

$$v(x, \lambda r) \leq C \lambda^D v(x, r), \quad \forall x \in M, \forall r \geq 0, \forall \lambda \geq 1, \quad (\text{D})$$

where $v(x, r) = \mu(B(x, r))$ denotes the volume of the ball $B(x, r)$ of center x and radius r . We also say that M is of homogeneous type. This property is equivalent to the existence of a constant $C > 0$ such that

$$v(x, 2r) \leq C v(x, r), \quad \forall x \in M, \forall r \geq 0.$$

Let Δ be the non-negative Laplace-Beltrami operator and let $p_t(x, y)$ be the heat kernel of M , that is, the kernel of the semigroup $(e^{-t\Delta})_{t \geq 0}$ acting on $L^2(M)$. We say that the heat kernel $p_t(x, y)$ satisfies a Gaussian upper bound if there exist constants $c, C > 0$ such that

$$p_t(x, y) \leq \frac{C}{v(x, \sqrt{t})} \exp(-c \frac{\rho^2(x, y)}{t}), \quad \forall t > 0, \forall x, y \in M. \quad (G)$$

Let $d(\Delta)^{-\frac{1}{2}}$ be the Riesz transform of the operator Δ where d denotes the exterior derivative on M . Since we have by integration by parts

$$\|d f\|_2 = \|\Delta^{\frac{1}{2}} f\|_2, \quad \forall f \in C_0^\infty(M),$$

the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ extends to a bounded operator from $L^2(M)$ to $L^2(\Lambda^1 T^* M)$, where $\Lambda^1 T^* M$ denotes the space of 1-forms on M . An interesting question is whether $d(\Delta)^{-\frac{1}{2}}$ can be extended to a bounded operator from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p \neq 2$. This problem has attracted attention in recent years. We recall some known results.

It was proved by Coulhon and Duong [13] that under the assumptions (D) and (G) , the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is of weak-type $(1, 1)$ and then bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1, 2]$. In addition, they gave an example of a complete non-compact Riemannian manifold satisfying (D) and (G) for which $d(\Delta)^{-\frac{1}{2}}$ is unbounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p > 2$. This manifold consists into two copies of \mathbb{R}^2 glued together around the unit circle. See also the article of Carron, Coulhon and Hassell [12] for further results on manifolds with Euclidean ends or the article of Guillarmou and Hassell [20] for complete non-compact and asymptotically conic Riemannian manifolds.

The counter-example in [13] shows that additional assumptions are needed to treat the case $p > 2$. In 2003, Coulhon and Duong [14] proved that if the manifold M satisfies (D) , (G) and the heat kernel $\vec{p}_t(x, y)$ associated with the Hodge-de Rham Laplacian $\vec{\Delta}$ acting on 1-forms satisfies a Gaussian upper bound, then the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1, \infty)$. The

proof is based on duality arguments and on the following estimate of the gradient of the heat kernel of M

$$|\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t} v(x, \sqrt{t})} e^{-c \frac{\rho^2(x, y)}{t}}, \forall x, y \in M, \forall t > 0,$$

which is a consequence of the relative Faber-Krahn inequalities satisfied by M and the Gaussian estimates satisfied by $e^{-t\vec{\Delta}}$.

In 1987, Bakry [5] proved that if the Ricci curvature is non-negative on M , then the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in (1, \infty)$. The proof uses probabilistic techniques and the domination

$$|e^{-t\vec{\Delta}} \omega| \leq e^{-t\Delta} |\omega|, \forall t > 0, \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^* M).$$

In this particular setting, (G) is satisfied, and hence the heat kernel $\vec{p}_t(x, y)$ satisfies a Gaussian upper bound too. Thus the result of Bakry can be recovered using the arguments of Coulhon and Duong [14]. Note that the result of Bakry does not contradict the counter-example of Coulhon and Duong since the gluing of two copies of \mathbb{R}^2 creates some negative curvature.

In 2004, Sikora [25] improved the previous result of Coulhon and Duong showing that if the manifold M satisfies (D) and the estimate

$$\|\vec{p}_t(x, \cdot)\|_{L^2}^2 \leq \frac{c}{v(x, \sqrt{t})}, \forall t > 0, \forall x \in M,$$

then the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $p \in [2, \infty)$. The proof is based on the method of the wave equation.

Auscher, Coulhon, Duong and Hofmann [4] characterized the boundedness of the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p > 2$ in terms of $L^p - L^p$ estimates of the gradient of the heat semigroup when the Riemannian manifold M satisfies Li-Yau estimates. More precisely, they proved that if $p_t(x, y)$ satisfies both Gaussian upper and lower bounds, then $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for $p \in [2, p_0)$ if and only if $\|d e^{-t\Delta}\|_{p-p} \leq \frac{C}{\sqrt{t}}$ for p in the same interval.

Inspired by [14], Devyver [17] proved a boundedness result for the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ in the setting of Riemannian manifolds satisfying a global Sobolev inequality of dimension N with an additional assumption that balls of great radius have a polynomial volume growth. It is known in this setting that both (D) and (G) are satisfied. He assumed that the negative part R_- of the Ricci curvature satisfies the

condition $R_- \in L^{\frac{N}{2}-\eta} \cap L^\infty$ for some $\eta > 0$ and that there is no harmonic 1-form on M . Under these assumptions, he showed that $\vec{p}_t(x, y)$ satisfies a Gaussian upper bound which implies the boundedness of the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, \infty)$. Without the assumption on harmonic 1-forms, it is also proved in [17] that $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, N)$.

In this article, we study the boundedness of the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for $p > 2$ assuming M satisfies the doubling volume property (D) and $p_t(x, y)$ satisfies a Gaussian upper bound (G). Before stating our results, we recall the Bochner formula $\vec{\Delta} = \nabla^* \nabla + R_+ - R_- =: H - R_-$, where R_+ (resp. R_-) is the positive part (resp. negative part) of the Ricci curvature and ∇ denotes the Levi-Civita connection on M . This formula allows us to consider the Hodge-de Rham Laplacian as a "generalized" Schrödinger operator acting on 1-forms. We then make a standard assumption on the negative part R_- ; namely, we suppose that R_- is ϵ -sub critical, which means that for a certain $\epsilon \in [0, 1)$

$$0 \leq (R_- \omega, \omega) \leq \epsilon (H \omega, \omega), \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M). \quad (\text{S-C})$$

For further information on condition (S-C), see the article of Coulhon and Zhang [15] and the references therein.

Under these assumptions, we prove the following results.

Theorem 1.1. *Assume that (D), (G) and (S-C) are satisfied. Then the Riesz transform $d^*(\vec{\Delta})^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^*M)$ to $L^p(M)$ and the Riesz transform $d(\vec{\Delta})^{-\frac{1}{2}}$ is bounded from $L^p(\Lambda^1 T^*M)$ to $L^p(\Lambda^2 T^*M)$ for all $p \in (p'_0, 2]$ where, $p'_0 = \left(\frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}\right)'$ if $D > 2$ and $p'_0 = 1$ if $D \leq 2$.*

Here and throughout this paper, p'_0 denotes the conjugate of p_0 .

Concerning the Riesz transform on functions, we have the following result.

Corollary 1.2. *Assume that (D), (G) and (S-C) are satisfied. Then the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, p_0)$ where, $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ if $D > 2$ and $p_0 = +\infty$ if $D \leq 2$.*

*In particular, the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, \frac{2D}{D-2})$ if $D > 2$ and all $p \in (1, +\infty)$ if $D \leq 2$.*

In these results, the constant D is as in (D) and ϵ is as in (S-C). Of course, we take the smallest possible D and ϵ for which (D) and (S-C) are satisfied. The operator

d denotes the exterior derivative acting from the space of 1-forms to the space of 2-forms or from the space of functions to the space of 1-forms according to the context. The operator d^* denotes the L^2 -adjoint of the exterior derivative d , the latter acting from the space of functions to the space of 1-forms.

Proof of Corollary 1.2. According to the commutation formula $\vec{\Delta}d = d\Delta$, we see that the adjoint operator of $d^*(\vec{\Delta})^{-\frac{1}{2}}$ is exactly $d(\Delta)^{-\frac{1}{2}}$. Then **Corollary 1.2** is an immediate consequence of **Theorem 1.1**. \square

Before stating our next result, we set

$$\text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) := \{\omega \in \mathcal{D}(\vec{\mathfrak{h}}) : \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M), (\omega, \vec{\Delta}\eta) = 0\},$$

where $\mathcal{D}(\vec{\mathfrak{h}})$ is the domain of the closed sesquilinear form \mathfrak{h} whose associated operator is H (see the next section for the definition of \mathfrak{h}). We prove the following.

Theorem 1.3. *Assume that both (D) and (G) are satisfied. In addition, suppose that for some $r_1, r_2 > 2$*

$$\int_0^1 \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_1}}} \right\| \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_2}}} \right\| \frac{dt}{\sqrt{t}} < +\infty \quad (1)$$

and

$$\text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\}. \quad (2)$$

Then there exists $\epsilon \in [0, 1)$ such that the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, p_0)$ where, $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ if $D > 2$ and $p_0 = +\infty$ if $D \leq 2$.

In particular, the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ for all $p \in (1, \frac{2D}{D-2})$ if $D > 2$ and all $p \in (1, +\infty)$ if $D \leq 2$.

We emphasize that in **Theorem 1.1**, **Corollary 1.2** and **Theorem 1.3**, neither a global Sobolev-type inequality nor any estimates on $\nabla_x p_t(x, y)$ or $\|\vec{p}_t(x, y)\|$ are assumed.

Condition (1) was introduced by Assaad and Ouhabaz [2]. Note that if $v(x, r) \simeq r^N$, then (1) means that $R_- \in L^{\frac{N}{2}-\eta} \cap L^{\frac{N}{2}+\eta}$ for some $\eta > 0$. In addition, we show that if the quantity

$$\|R_-^{\frac{1}{2}}\|_{vol} := \int_0^1 \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_1}}} \right\| \frac{dt}{\sqrt{t}} + \int_1^\infty \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_2}}} \right\| \frac{dt}{\sqrt{t}}$$

is small enough, then R_- is ϵ -sub-critical for some $\epsilon \in [0, 1)$ depending on $\|R_-^{\frac{1}{2}}\|_{vol}$ and on the constants appearing in (D) and (G).

Condition (2) was also considered by Devyver [17]. Under our assumptions, the space $Ker_{\mathcal{D}(\vec{\Delta})}(\vec{\Delta})$ is precisely the space of L^2 harmonic 1-forms. See the last section for more details.

The proof of **Theorem 1.1** uses similar technics as in Assaad and Ouhabaz [2] where the Riesz transforms of Schrödinger operators $-\Delta + V$ are studied for signed potentials. In our setting, $\vec{\Delta} = \nabla^* \nabla + R_+ - R_-$ can be seen as a "generalized" Schrödinger operator. However the arguments from [2] need substantial modifications, since our Schrödinger operator is a vector-valued operator. In particular we cannot use any sub-Markovian property, as is used in [2].

In Section 2, we discuss some preliminaries which are necessary for the main proofs. In Section 3, we prove that under the assumptions (D), (G) and (S-C), the operator $\vec{\Delta}$ generates a uniformly bounded \mathcal{C}^0 -semigroup on $L^p(\Lambda^1 T^* M)$ for all $p \in (p'_0, p_0)$ where p_0 is as in **Theorem 1.1**. Section 4 is devoted to the proof of **Theorem 1.1**. Here we use the results of Section 3. In the last section we prove **Theorem 1.3** ; one of the main ingredient is to prove that if the manifold M satisfies condition (1), then R_- satisfies (S-C) if and only if condition (2) is satisfied. Here the constant ϵ appearing in (S-C) is the L^2 - L^2 norm of the operator $H^{-\frac{1}{2}} R_- H^{-\frac{1}{2}}$.

2 Preliminaries

For all $x \in M$ we denote by $\langle \cdot, \cdot \rangle_x$ the inner product in the tangent space $T_x M$, in the cotangent space $T_x^* M$ or in the tensor product $T_x^* M \otimes T_x^* M$. By (\cdot, \cdot) we denote the inner product in the Lebesgue space $L^2(M)$ of functions, in the Lebesgue space $L^2(\Lambda^1 T^* M)$ of 1-forms or in the Lebesgue space $L^2(\Lambda^2 T^* M)$ of 2-forms. By $\|\cdot\|_p$ we denote the usual norm in $L^p(M)$, $L^p(\Lambda^1 T^* M)$ or $L^p(\Lambda^2 T^* M)$ and by $\|\cdot\|_{p \rightarrow q}$ the norm of operators from L^p to L^q (according to the context). The spaces $\mathcal{C}_0^\infty(M)$ and $\mathcal{C}_0^\infty(\Lambda^1 T^* M)$ denote respectively the space of smooth functions and smooth 1-forms with compact support on M . We denote by d the exterior derivative on M and d^* its L^2 -adjoint operator. According to the context, the operator d acts from the space of functions on M to $\Lambda^1 T^* M$ or from $\Lambda^1 T^* M$ to $\Lambda^2 T^* M$. If E is a subset of M , χ_E denotes the indicator function of E .

For $\omega, \eta \in \Lambda^1 T^* M$ and for $x \in M$, we denote by $\omega(x) \otimes \eta(x)$ the tensor product of the linear forms $\omega(x)$ and $\eta(x)$. The inner product on the cotangent space $T_x^* M$

induces an inner product on each tensor product $T_x^*M \otimes T_x^*M$ given by

$$\langle \omega_1(x) \otimes \eta_1(x), \omega_2(x) \otimes \eta_2(x) \rangle_x = \langle \omega_1(x), \omega_2(x) \rangle_x \langle \eta_1(x), \eta_2(x) \rangle_x,$$

for all $\omega_1, \omega_2, \eta_1, \eta_2 \in \Lambda^1 T^*M$ and $x \in M$.

We consider Δ the non-negative Laplace-Beltrami operator acting on $L^2(M)$ and $p_t(x, y)$ the heat kernel of M , that is, the integral kernel of the semigroup $e^{-t\Delta}$.

We consider the Hodge-de Rham Laplacian $\vec{\Delta} = d^*d + dd^*$ acting on $L^2(\Lambda^1 T^*M)$. The Bochner formula says that $\vec{\Delta} = \nabla^* \nabla + R_+ - R_-$, where R_+ (resp. R_-) is the positive part (resp. negative part) of the Ricci curvature and ∇ denotes the Levi-Civita connection on M . It allows us to look at $\vec{\Delta}$ as a "generalized" Schrödinger operator with signed vector potential $R_+ - R_-$.

We define the self-adjoint operator $H = \nabla^* \nabla + R_+$ on $L^2(\Lambda^1 T^*M)$ using the method of sesquilinear forms. That is, for all $\omega, \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$, we set

$$\vec{\mathfrak{h}}(\omega, \eta) = \int_M \langle \nabla \omega(x), \nabla \eta(x) \rangle_x d\mu + \int_M \langle R_+(x) \omega(x), \eta(x) \rangle_x d\mu,$$

$$\text{and } \mathcal{D}(\vec{\mathfrak{h}}) = \overline{\mathcal{C}_0^\infty(\Lambda^1 T^*M)}^{\|\cdot\|_{\vec{\mathfrak{h}}}},$$

where $\|\omega\|_{\vec{\mathfrak{h}}} = \sqrt{\vec{\mathfrak{h}}(\omega, \omega) + \|\omega\|_2^2}$.

We say that R_- is ϵ -sub-critical if for a certain constant $0 \leq \epsilon < 1$

$$0 \leq (R_- \omega, \omega) \leq \epsilon (H \omega, \omega), \forall \omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M). \quad (\text{S-C})$$

Under the assumption (S-C), we define the self-adjoint operator $\vec{\Delta} = \nabla^* \nabla + R_+ - R_-$ on $L^2(\Lambda^1 T^*M)$ as the operator associated with the form

$$\vec{\mathfrak{a}}(\omega, \eta) = \vec{\mathfrak{h}}(\omega, \eta) - \int_M \langle R_-(x) \omega(x), \eta(x) \rangle_x d\mu,$$

$$\mathcal{D}(\vec{\mathfrak{a}}) = \mathcal{D}(\vec{\mathfrak{h}}).$$

It is well known by the KLMN theorem (see [23], Theorem 1.19, p.12) that $\vec{\mathfrak{a}}$ is a closed form, bounded from below. Therefore it has an associated self-adjoint operator which is $H - R_-$.

In order to use the technics in [2], we need first to prove that the semigroup $(e^{-t\vec{\Delta}})_{t \geq 0}$ is uniformly bounded on $L^p(\Lambda^1 T^*M)$ for all $p \in (p'_0, 2]$.

3 L^p theory of the heat semigroup on forms

To study the boundedness of the semigroup $(e^{-t\vec{\Delta}})_{t \geq 0}$ on $L^p(\Lambda^1 T^*M)$ for $p \neq 2$, we use perturbation arguments as in [22], where Liskevich and Semenov studied semigroups associated with Schrödinger operators with negative potentials. The main result of this section is the following.

Theorem 3.1. *Suppose that the assumptions (D) , (G) and $(S-C)$ are satisfied. Then the operator $\vec{\Delta} = \nabla^* \nabla + R_+ - R_-$ generates a uniformly bounded \mathcal{C}^0 -semigroup on $L^p(\Lambda^1 T^*M)$ for all $p \in (p'_0, p_0)$ where $p_0 = \frac{2D}{(D-2)(1-\sqrt{1-\epsilon})}$ if $D > 2$ and $p_0 = +\infty$ if $D \leq 2$.*

To demonstrate **Theorem 3.1** we proceed in two steps. The first step consists in proving the result for p in the smaller range $[p'_1, p_1]$ where $p_1 = \frac{2}{1-\sqrt{1-\epsilon}}$; we do this in **Proposition 3.3**, with the help of **Lemma 3.2** below. The second step consists in extending this interval using interpolation between the estimates of **Proposition 3.5** and **Proposition 3.6**.

We begin with the following lemma.

Lemma 3.2. *Let $p \geq 1$. For any suitable $\omega \in \Lambda^1 T^*M$ and for every $x \in M$*

$$\langle \nabla(\omega|\omega|^{p-2})(x), \nabla\omega(x) \rangle_x \geq \frac{4(p-1)}{p^2} \langle \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) \rangle_x. \quad (3)$$

Remark 1. In the previous statement, "suitable" means that the calculations make sense with such a ω . For instance, a form $\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$ is suitable.

Proof. To make the calculations simpler, for every $x \in M$, we work in a synchronous frame. That is we choose an orthonormal frame $\{X_i\}_i$ to have the Christoffel symbols $\Gamma_{ij}^k(x) = 0$ at x (see for instance [18] p.93 or [24] p.70,73 for more details). In what follows, we use properties satisfied by the Levi-Civita connection ∇ , which can be found in [24] p.64-66.

Considering $\{\theta^i\}_i$ the orthonormal frame of 1-forms dual to $\{X_i\}_i$, we write for a 1-form ω , $\omega(y) = \sum_i f_i(y)\theta^i = \sum_i \omega_i(y)$ for all y in a neighborhood of x . With this

choice of local coordinates we have at x , $|\omega(x)|_x = \sqrt{\sum_i f_i(x)^2}$ and $\nabla\theta^i = 0$ for all i .

Then, when $\omega(x) \neq 0$, we obtain

$$\nabla(|\omega|)(x) = \frac{\sum_i f_i(x) df_i(x)}{|\omega(x)|_x}. \quad (4)$$

We recall that we have an inner product in each tensor product $T_x^*M \otimes T_x^*M$ satisfying

$$\langle \omega_1(x) \otimes \eta_1(x), \omega_2(x) \otimes \eta_2(x) \rangle_x = \langle \omega_1(x), \omega_2(x) \rangle_x \langle \eta_1(x), \eta_2(x) \rangle_x, \quad (5)$$

for all $\omega_1, \omega_2, \eta_1, \eta_2 \in \Lambda^1 T^*M$ and $x \in M$. In particular for all $\omega, \eta \in \Lambda^1 T^*M$ and $x \in M$

$$|\omega(x) \otimes \eta(x)|_x = |\omega(x)|_x |\eta(x)|_x. \quad (6)$$

To avoid dividing by 0, one can replace $|\omega(x)|_x$ by $|\omega(x)|_{x,\epsilon} := \sqrt{\sum_i f_i(x)^2 + \epsilon}$ for some $\epsilon > 0$, make the calculations and let ϵ tend to 0. For simplicity, we ignore this step and make the calculations formally.

We first deal with the RHS of (3). Using (4) and (6), we have

$$\begin{aligned} & \langle \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) \rangle_x \\ &= \left| |\omega(x)|_x^{\frac{p}{2}-1} \nabla \omega(x) + \left(\frac{p}{2} - 1\right) |\omega(x)|_x^{\frac{p}{2}-3} \left(\sum_i f_i(x) df_i(x)\right) \otimes \omega(x) \right|_x^2 \\ &= |\omega(x)|_x^{p-2} |\nabla \omega(x)|_x^2 + \left(\frac{p}{2} - 1\right)^2 |\omega(x)|_x^{p-6} \left|\sum_i f_i(x) df_i(x)\right|_x^2 |\omega(x)|_x^2 \\ &\quad + (p-2) |\omega(x)|_x^{p-4} \langle \nabla \omega(x), \left(\sum_i f_i(x) df_i(x)\right) \otimes \omega(x) \rangle_x. \end{aligned}$$

Now noticing that $(\theta^i)_i$ is an orthonormal basis of T_x^*M and using (5) yield

$$\begin{aligned} \langle \nabla \omega(x), \left(\sum_i f_i(x) df_i(x)\right) \otimes \omega(x) \rangle_x &= \langle \sum_j df_j(x) \otimes \theta^j, \left(\sum_i f_i(x) df_i(x)\right) \otimes \omega(x) \rangle_x \\ &= \sum_{i,k} f_i(x) f_k(x) \langle df_i(x), df_k(x) \rangle_x \\ &= \left|\sum_i f_i(x) df_i(x)\right|_x^2. \end{aligned}$$

Then we obtain

$$\begin{aligned} & \langle \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) \rangle_x \\ &= |\omega(x)|_x^{p-2} |\nabla \omega(x)|_x^2 + \left(\frac{p^2}{4} - 1\right) |\omega(x)|_x^{p-4} \left|\sum_i f_i(x) df_i(x)\right|_x^2. \end{aligned}$$

Using the equality $|\nabla \omega(x)|_x^2 = \sum_i |df_i(x)|_x^2$ at x , a simple calculation gives for all i

$$\begin{aligned} \left|\sum_i f_i(x) df_i(x)\right|_x^2 &= \sum_i f_i(x)^2 |df_i(x)|_x^2 + 2 \sum_{i < j} f_i(x) f_j(x) \langle df_i(x), df_j(x) \rangle_x \\ &= |\omega(x)|_x^2 |\nabla \omega(x)|_x^2 - \sum_i \sum_{j \neq i} f_j(x)^2 |df_i(x)|_x^2 + 2 \sum_{i < j} f_i(x) f_j(x) \langle df_i(x), df_j(x) \rangle_x. \end{aligned}$$

Thus for all i

$$|\sum_i f_i(x) df_i(x)|_x^2 = |\omega(x)|_x^2 |\nabla \omega(x)|_x^2 - \sum_{i < j} |f_i(x) df_j(x) - f_j(x) df_i(x)|_x^2. \quad (7)$$

Finally we obtain

$$\begin{aligned} & < \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) >_x \\ &= \frac{p^2}{4} |\omega(x)|_x^2 |\nabla \omega(x)|_x^2 - (\frac{p^2}{4} - 1) |\omega(x)|_x^{p-4} \sum_{i < j} |f_i(x) df_j(x) - f_j(x) df_i(x)|_x^2. \end{aligned}$$

Let us deal with the LHS of (3) now. We write

$$< \nabla(\omega|\omega|^{p-2})(x), \nabla \omega(x) >_x = \sum_i < \nabla(\omega_i|\omega|^{p-2})(x), \nabla \omega(x) >_x.$$

Using again (5), we observe that for all i, j with $i \neq j$, $< \nabla \omega_i(x), \nabla \omega_j(x) >_x = 0$. Thus, using (4), we obtain that for all i

$$\begin{aligned} & < \nabla(\omega_i|\omega|^{p-2})(x), \nabla \omega(x) >_x \\ &= |\omega(x)|_x^{p-2} |\nabla \omega_i(x)|_x^2 + (p-2) |\omega(x)|_x^{p-4} \sum_j f_j(x) < df_j(x) \otimes \omega_i(x), \nabla \omega(x) >_x. \end{aligned}$$

From (5) again, we deduce that for all i, j

$$\begin{aligned} < df_j(x) \otimes \omega_i(x), \nabla \omega(x) >_x &= f_i(x) < df_j(x) \otimes \theta^i, \sum_k df_k(x) \otimes \theta^k >_x \\ &= f_i(x) < df_i(x), df_j(x) >_x. \end{aligned}$$

Hence for all i

$$\begin{aligned} & < \nabla(\omega_i|\omega|^{p-2})(x), \nabla \omega(x) >_x \\ &= |\omega(x)|_x^{p-2} |\nabla \omega_i(x)|_x^2 + (p-2) |\omega(x)|_x^{p-4} \sum_j f_i(x) f_j(x) < df_i(x), df_j(x) >_x. \end{aligned}$$

As we did before to obtain (7), we find

$$\begin{aligned} & < \nabla(\omega|\omega|^{p-2})(x), \nabla \omega(x) >_x \\ &= \sum_i < \nabla(\omega_i|\omega|^{p-2})(x), \nabla \omega(x) >_x \\ &= (p-1) |\omega(x)|_x^{p-2} |\nabla \omega(x)|_x^2 - (p-2) |\omega(x)|_x^{p-4} \sum_{i < j} |f_i(x) df_j(x) - f_j(x) df_i(x)|_x^2. \end{aligned}$$

To conclude we calculate

$$\begin{aligned}
\frac{1}{p-1} &< \nabla(\omega|\omega|^{p-2})(x), \nabla\omega(x) >_x - \frac{4}{p^2} < \nabla(\omega|\omega|^{\frac{p}{2}-1})(x), \nabla(\omega|\omega|^{\frac{p}{2}-1})(x) >_x \\
&= \left(\frac{4}{p^2} \left(\frac{p^2}{4} - 1 \right) - \frac{p-2}{p-1} \right) |\omega(x)|_x^{p-4} \sum_{i < j} |f_i(x)df_j(x) - f_j(x)df_i(x)|_x^2 \\
&= \frac{(p-2)^2}{(p-1)p^2} |\omega(x)|_x^{p-4} \sum_{i < j} |f_i(x)df_j(x) - f_j(x)df_i(x)|_x^2 \\
&\geq 0.
\end{aligned}$$

This proves the lemma. □

We are now able to prove that the semigroup $(e^{-t\vec{\Delta}})_{t \geq 0}$ is uniformly bounded on $L^p(\Lambda^1 T^* M)$ for some $p \neq 2$ under the assumption (S-C).

Proposition 3.3. *Suppose that the negative part R_- of the Ricci curvature satisfies the assumption (S-C). Then the operator $\vec{\Delta}$ generates a \mathcal{C}^0 -semigroup of contractions on $L^p(\Lambda^1 T^* M)$ for all $p \in [p'_1, p_1]$ where $p_1 = \frac{2}{1-\sqrt{1-\epsilon}}$.*

Proof. We consider $\eta \in \mathcal{C}_0^\infty(\Lambda^1 T^* M)$ and set $\omega_t = e^{-t\vec{\Delta}}\eta$ for all $t \geq 0$. Taking the inner product of both sides of the equation $-\frac{d}{dt}\omega_t = \vec{\Delta}\omega_t$ with $|\omega_t|^{p-2}\omega_t$ and integrating over M yield

$$\begin{aligned}
-\frac{1}{p} \frac{d}{dt} \|\omega_t\|_p^p &= (\vec{\Delta}\omega_t, |\omega_t|^{p-2}\omega_t) \\
&= \int_M \langle \nabla\omega_t(x), \nabla(|\omega_t|^{p-2}\omega_t)(x) \rangle_x d\mu + ((R_+ - R_-)\omega_t, |\omega_t|^{p-2}\omega_t).
\end{aligned}$$

Since we have by linearity of $R_+(x)$ and $R_-(x)$

$$((R_+ - R_-)\omega_t, |\omega_t|^{p-2}\omega_t) = ((R_+ - R_-)(|\omega_t|^{\frac{p}{2}-1}\omega_t), |\omega_t|^{\frac{p}{2}-1}\omega_t),$$

the previous lemma and the the assumption (S-C) yield

$$-\frac{1}{p} \frac{d}{dt} \|\omega_t\|_p^p \geq \left(\frac{4(p-1)}{p^2} - \epsilon \right) \|H^{\frac{1}{2}}(|\omega_t|^{\frac{p}{2}-1}\omega_t)\|_2^2.$$

Then for all $p \in [\frac{2}{1+\sqrt{1-\epsilon}}, \frac{2}{1-\sqrt{1-\epsilon}}]$

$$-\frac{1}{p} \frac{d}{dt} \|\omega_t\|_p^p \geq 0.$$

Therefore $\|\omega_t\|_p \leq \|\omega_0\|_p$, that is,

$$\|e^{-t\vec{\Delta}}\eta\|_p \leq \|\eta\|_p, \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M),$$

and we conclude by a usual density argument. \square

Actually, as in [22] and [2], we can obtain a better interval than $[p'_1, p_1]$ by interpolation arguments and prove **Theorem 3.1**. The ideas of this proof are the same as in [2]. However we give some details which we adapt to our setting.

Lemma 3.4. *Let q be such that $2 < q \leq \infty$ and $\frac{q-2}{q}D < 2$. Then for all $x \in M$, $t > 0$ and $\omega \in \mathcal{D}(\vec{\alpha})$*

$$\|\chi_{B(x, \sqrt{t})}\omega\|_q \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2}-\frac{1}{q}}} \left(\|\omega\|_2 + \sqrt{t} \|\vec{\Delta}^{\frac{1}{2}}\omega\|_2 \right).$$

Proof. We recall that H denotes the operator $\nabla^* \nabla + R_+$ and that we have the domination $|e^{-tH}\omega| \leq e^{-t\Delta}|\omega|$ for any $\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$ (see [6] p.171,172). Since we assume (G), the heat kernel $p_t^H(x, y)$ associated to the semigroup $(e^{-tH})_{t \geq 0}$ satisfies a Gaussian upper bound

$$\|p_t^H(x, y)\| \leq \frac{C}{v(x, \sqrt{t})} \exp(-c \frac{\rho^2(x, y)}{t}), \forall t > 0, \forall x, y \in M. \quad (8)$$

From (8) and the doubling volume property (D), it is not difficult to show that for all $x \in M$ and $0 < s \leq t$

$$\|\chi_{B(x, \sqrt{t})}e^{-sH}\|_{2-\infty} \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2}}} \left(\frac{t}{s} \right)^{\frac{D}{4}}. \quad (9)$$

Indeed for $x \in M$, $y \in B(x, \sqrt{t})$ and $0 < s \leq t$, the inclusion of balls

$$B(x, \sqrt{t}) \subset B(y, \sqrt{t} + \rho(x, y)) \subset B(y, 2\sqrt{t})$$

and the doubling volume property yield

$$v(x, \sqrt{t}) \leq C \left(\frac{t}{s} \right)^{\frac{D}{2}} v(y, \sqrt{s}). \quad (10)$$

In addition (8) implies that for all $x \in M$, $y \in B(x, \sqrt{t})$, $\omega \in L^2(\Lambda^1 T^*M)$ and $0 < s \leq t$

$$|\chi_{B(x, \sqrt{t})}(y)e^{-sH}\omega(y)| \leq \int_M \frac{C}{v(y, \sqrt{s})} \exp(-c \frac{\rho^2(y, z)}{s}) |\omega(z)|_z d\mu(z).$$

Writing $v(y, \sqrt{s}) = v(y, \sqrt{s})^{\frac{1}{2}} v(y, \sqrt{s})^{\frac{1}{2}}$, then using (10) and the Hölder inequality, leads to

$$|\chi_{B(x, \sqrt{t})}(y) e^{-sH} \omega(y)| \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2}}} \left(\frac{t}{s} \right)^{\frac{D}{4}} \left(\int_M \frac{\exp(-2c \frac{\rho^2(y, z)}{s})}{v(y, \sqrt{s})} d\mu(z) \right)^{\frac{1}{2}} \|\omega\|_2. \quad (11)$$

We use a standard decomposition of M into annuli to obtain

$$\begin{aligned} \int_M \exp(-2c \frac{\rho^2(y, z)}{s}) d\mu(z) &\leq \sum_{k=0}^{\infty} \int_{k\sqrt{s} \leq \rho(y, z) \leq (k+1)\sqrt{s}} \exp(-2ck^2) d\mu(z) \\ &\leq \sum_{k=0}^{\infty} \exp(-2ck^2) v(y, (k+1)\sqrt{s}). \end{aligned}$$

Then the doubling volume property (D) implies

$$\int_M \exp(-2c \frac{\rho^2(y, z)}{s}) d\mu(z) \leq C v(y, \sqrt{s}). \quad (12)$$

We deduce (9) from (11) and (12).

Now since the semigroup $(e^{-tH})_{t \geq 0}$ is bounded on $L^2(\Lambda^1 T^* M)$, it follows by interpolation that

$$\|\chi_{B(x, \sqrt{t})} e^{-sH}\|_{2-q} \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2}-\frac{1}{q}}} \left(\frac{t}{s} \right)^{\frac{D}{2}(\frac{1}{2}-\frac{1}{q})}, \quad (13)$$

for all $2 < q \leq \infty$. Note that since the semigroup $(e^{-tH})_{t \geq 0}$ is analytic on $L^2(\Lambda^1 T^* M)$, we have for all $\omega \in L^2(\Lambda^1 T^* M)$ and all $s \geq 0$

$$\|H^{\frac{1}{2}} e^{-sH} \omega\|_2 \leq \frac{C}{\sqrt{s}} \|\omega\|_2. \quad (14)$$

Then writing for all $\omega \in \mathcal{D}(\vec{\mathfrak{a}})$

$$\omega = e^{-tH} \omega + \int_0^t H e^{-sH} \omega ds = e^{-tH} \omega + \int_0^t e^{-\frac{s}{2}H} H^{\frac{1}{2}} e^{-\frac{s}{2}H} H^{\frac{1}{2}} \omega ds,$$

and using (13) and (14), we obtain

$$\|\chi_{B(x, \sqrt{t})} \omega\|_q \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2}-\frac{1}{q}}} \left(\|\omega\|_2 + t^{\frac{D}{2}(\frac{1}{2}-\frac{1}{q})} \|H^{\frac{1}{2}} \omega\|_2 \int_0^t s^{-\frac{1}{2}-\frac{D}{2}(\frac{1}{2}-\frac{1}{q})} ds \right).$$

The convergence of the last integral is ensured for q such that $\frac{q-2}{q}D < 2$ and we then have for such q

$$\|\chi_{B(x, \sqrt{t})}\omega\|_q \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{2}-\frac{1}{q}}} \left(\|\omega\|_2 + \sqrt{t}\|H^{\frac{1}{2}}\omega\|_2 \right). \quad (15)$$

To conclude the proof, we need to have the estimate (15) with the operator $\vec{\Delta}$ instead of H . This is a consequence of the assumption (S-C) since we have for all $\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^* M)$, $\|H^{\frac{1}{2}}\omega\|_2^2 \leq \frac{1}{1-\epsilon} \|\vec{\Delta}^{\frac{1}{2}}\omega\|_2^2$. \square

Remark 2. **Lemma 3.4** also follows from [10], Proposition 2.3.1 since the heat kernel of H satisfies a Gaussian estimate.

A key result to obtain **Theorem 3.1** is the following proposition.

Proposition 3.5. *We consider $2 \leq p < p_1$ and q such that $1 \leq q \leq \infty$ and $\frac{q-1}{q}D < 2$. Then for all $x \in M$ and $t > 0$*

$$\|\chi_{B(x, \sqrt{t})}e^{-s\vec{\Delta}}\|_{p-pq} \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{p}-\frac{1}{pq}}} \left(\max \left(1, \sqrt{\frac{t}{s}} \right) \right)^{\frac{2}{p}}.$$

Proof. Combining **Lemma 3.4**, **Proposition 3.3** and following the proof of Proposition 2.2 from [2] lead to the desired result. \square

Following the ideas in [2], the last property we need to check is that the semigroup $(e^{-t\vec{\Delta}})_{t \geq 0}$ satisfies the Davies-Gaffney estimates (also called L^2 - L^2 off-diagonal estimates in [2]). This is the purpose of the next proposition. Its proof is based on the well-known Davies' perturbation method. Another proof can be found in [25], Theorem 6.

Proposition 3.6. *Let E, F be two closed subsets of M . For any $\eta \in L^2(\Lambda^1 T^* M)$ with support in E*

$$\|e^{-t\vec{\Delta}}\eta\|_{L^2(F)} \leq e^{-\frac{\rho^2(E, F)}{2t}} \|\eta\|_2.$$

Proof. We choose a constant $\alpha > 0$ and a bounded Lipschitz function ϕ such that $|\nabla\phi(x)|_x \leq 1$ for almost every $x \in M$. We define the operator $\vec{\Delta}_\alpha = e^{\alpha\phi}\vec{\Delta}e^{-\alpha\phi}$ with the sesquilinear form

$$\vec{a}_\alpha(u, v) = \vec{a}(e^{-\alpha\phi}u, e^{\alpha\phi}v), \quad \mathcal{D}(\vec{a}_\alpha) = \mathcal{D}(\vec{a}).$$

Note that since ϕ is bounded then $e^{\pm\alpha\phi}u \in \mathcal{D}(\vec{a})$ for all $u \in \mathcal{D}(\vec{a})$.

For $\omega \in \mathcal{D}(\vec{a})$, we have

$$\begin{aligned}
& ((\vec{\Delta}_\alpha + \alpha^2)\omega, \omega) \\
&= \int_M \langle \nabla(e^{-\alpha\phi}\omega)(x), \nabla(e^{\alpha\phi}\omega)(x) \rangle_x d\mu + ((R_+ - R_-)\omega, \omega) + \alpha^2 \|\omega\|_2^2 \\
&= \int_M \langle e^{-\alpha\phi(x)} \nabla\omega(x) - \alpha e^{-\alpha\phi(x)} \nabla\phi(x) \otimes \omega(x), \\
&\quad e^{\alpha\phi(x)} \nabla\omega(x) + \alpha e^{\alpha\phi(x)} \nabla\phi(x) \otimes \omega(x) \rangle_x d\mu \\
&\quad + ((R_+ - R_-)\omega, \omega) + \alpha^2 \|\omega\|_2^2 \\
&= \|H^{\frac{1}{2}}\omega\|_2^2 - \alpha^2 \int_M |\nabla\phi(x)|_x^2 |\omega(x)|_x^2 d\mu - (R_- \omega, \omega) + \alpha^2 \|\omega\|_x^2 \\
&\geq 0.
\end{aligned}$$

The last inequality follows from the fact that the operator $\vec{\Delta}$ is non-negative and $|\nabla\phi(x)| \leq 1$ for almost every $x \in M$. As a consequence, the operator $\vec{\Delta}_\alpha + \alpha^2$ is positive and self-adjoint on $L^2(\Lambda^1 T^*M)$ and then $-(\vec{\Delta}_\alpha + \alpha^2)$ generates a \mathcal{C}^0 -semigroup of contractions on $L^2(\Lambda^1 T^*M)$. Therefore for all $\eta \in L^2(\Lambda^1 T^*M)$

$$\|e^{-t\vec{\Delta}_\alpha}\eta\|_2 \leq e^{t\alpha^2} \|\eta\|_2.$$

Now we consider E and F two closed subsets of M , $\eta \in L^2(\Lambda^1 T^*M)$ with support in E and $\phi_k(x) := \min(\rho(x, E), k)$ for $k \in \mathbb{N}$. Since $e^{\alpha\phi_k}\eta = \eta$, we have $e^{-t\vec{\Delta}}\eta = e^{-\alpha\phi_k}e^{-t\vec{\Delta}_\alpha}\eta$. Thus we obtain

$$\|e^{-t\vec{\Delta}}\eta\|_{L^2(F)} \leq e^{-\alpha \min(\rho(E, F), k)} e^{t\alpha^2} \|\eta\|_2.$$

To end the proof, let k tends to infinity and set $\alpha = \frac{\rho(E, F)}{2t}$.

□

Finally we give the proof of **Theorem 3.1**.

Proof of Theorem 3.1. For $x \in M$, $t \geq 0$ and $k \in \mathbb{N}$, we denote by $A(x, \sqrt{t}, k)$ the annulus $B(x, (k+1)\sqrt{t}) \setminus B(x, k\sqrt{t})$. Noticing that

$$\|\chi_{B(x, \sqrt{t})} e^{-t\vec{\Delta}} \chi_{A(x, \sqrt{t}, k)}\|_{p-pq} \leq \|\chi_{B(x, \sqrt{t})} e^{-t\vec{\Delta}}\|_{p-pq},$$

and using **Proposition 3.5**, we obtain the estimate

$$\|\chi_{B(x, \sqrt{t})} e^{-t\vec{\Delta}} \chi_{A(x, \sqrt{t}, k)}\|_{p-pq} \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{p} - \frac{1}{pq}}}, \quad (16)$$

for all $p \in [2, p_1)$ and q such that $1 \leq q \leq \infty$ and $\frac{q-1}{q}D < 2$. Interpolating (16) with the Davies-Gaffney estimate of **Proposition 3.6** yields

$$\|\chi_{B(x, \sqrt{t})} e^{-t\vec{\Delta}} \chi_{A(x, \sqrt{t}, k)}\|_{r-s} \leq \frac{C}{v(x, \sqrt{t})^{\frac{1}{r}-\frac{1}{s}}} e^{-ck^2},$$

for all $r \in [2, p_1)$ and all $s \in (2, p_1 q_0)$ where $q_0 = +\infty$ if $D \leq 2$ and $q_0 = \frac{D}{D-2}$ if $D > 2$. Since the semigroup $(e^{-t\vec{\Delta}})_{t \geq 0}$ is analytic on $L^2(\Lambda^1 T^* M)$ and uniformly bounded on $L^p(\Lambda^1 T^* M)$ for all $p \in [p'_1, p_1]$, Proposition 3.12 in [23] ensures that it is analytic on $L^p(\Lambda^1 T^* M)$ for all $p \in (p'_1, p_1)$. Therefore applying [7] Theorem 1.1, we deduce that $(e^{-t\vec{\Delta}})_{t \geq 0}$ is bounded analytic on $L^p(\Lambda^1 T^* M)$ for all $p \in [2, p_1 q_0) = [2, p_0)$. The case $p \in (p'_0, 2]$ is obtained by a usual duality argument. \square

4 Proof of Theorem 1.1

We start with the following L^p - L^q off-diagonal estimates for the semigroup $(e^{-t\vec{\Delta}})_{t \geq 0}$, which are consequences of the results of the previous section.

Theorem 4.1. *Suppose that (D), (G) and (S-C) are satisfied. Then for all $r, t > 0$, $x, y \in M$ and all $p \in (p'_0, p_0)$, $q \in [p, p_0)$*

$$(i) \quad \|\chi_{B(x, r)} e^{-t\vec{\Delta}} \chi_{B(y, r)}\|_{p-q} \leq \frac{C}{v(x, r)^{\frac{1}{p}-\frac{1}{q}}} \left(\max\left(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}\right) \right)^\beta e^{-c \frac{\rho^2(B(x, r), B(y, r))}{t}},$$

$$(ii) \quad \|\chi_{C_j(x, r)} e^{-t\vec{\Delta}} \chi_{B(x, r)}\|_{p-q} \leq \frac{C e^{-c \frac{4^j r^2}{t}}}{v(x, r)^{\frac{1}{p}-\frac{1}{q}}} \left(\max\left(\frac{2^{j+1} r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1} r}\right) \right)^\beta,$$

where $C_j(x, r) = B(x, 2^{j+1} r) \setminus B(x, 2^j r)$ and $\beta \geq 0$ depends on p and q .

Proof. We first treat the case $p \geq 2$.

We recall that from **Proposition 3.6**, we have for E and F two closed subsets of M

$$\|\chi_F e^{-t\vec{\Delta}} \chi_E\|_{2-2} \leq e^{-\frac{\rho^2(E, F)}{2t}}, \quad (17)$$

and from **Theorem 3.1**, we have for all $p \in (p'_0, p_0)$

$$\|e^{-t\vec{\Delta}}\|_{p-p} \leq C. \quad (18)$$

Using the Riesz-Thorin interpolation theorem from (17) and (18) implies the L^p - L^p off-diagonal estimate

$$\|\chi_F e^{-t\vec{\Delta}} \chi_E\|_{p-p} \leq C e^{-c \frac{\rho^2(E,F)}{t}}, \quad (19)$$

for all $t \geq 0$ and $p \in (p'_0, p_0)$. Taking $p \in [2, p_1)$ and using interpolation from (19) and **Proposition 3.5** yield

$$\|\chi_{B(x,r)} e^{-t\vec{\Delta}} \chi_{B(y,r)}\|_{p-pu} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left[\max(1, \frac{r}{\sqrt{t}}) \right]^\beta e^{-c \frac{\rho^2(B(x,r), B(y,r))}{t}},$$

for $p \in [2, p_1)$ and $u \in [1, \infty)$ if $D \leq 2$ or $u \in [1, \frac{D}{D-2})$ if $D > 2$. Here β is a non-negative constant depending on p and u .

If $D \leq 2$, we have the L^2 - L^q off-diagonal estimate for all $q \in [2, +\infty)$.

If $D > 2$, we can deduce, by a composition argument, L^2 - L^q off-diagonal estimates for $q \in [2, p_0)$ from L^2 - L^p and L^p - L^{pu} off-diagonal estimates with $p \in [2, p_1)$ and $u \in [1, \frac{D}{D-2})$. More precisely, we obtain

$$\|\chi_{B(x,r)} e^{-t\vec{\Delta}} \chi_{B(y,r)}\|_{p-pu} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left[\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}) \right]^\beta e^{-c \frac{\rho^2(B(x,r), B(y,r))}{t}},$$

for all $2 \leq p \leq q < p_0$.

The case $p'_0 < p \leq q \leq 2$ is obtained by duality and composition arguments. More precisely, we obtain

$$\|\chi_{B(x,r)} e^{-t\vec{\Delta}} \chi_{B(y,r)}\|_{p-pu} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left[\max(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}) \right]^\beta e^{-c \frac{\rho^2(B(x,r), B(y,r))}{t}},$$

for all $p'_0 < p \leq q < p_0$, which is (i). The reader can find more details in [2] Theorem 2.6.

Now we prove (ii). Writing

$$\chi_{C_j(x,r)} e^{-t\vec{\Delta}} \chi_{B(x,r)} = \chi_{C_j(x,r)} \chi_{B(x, 2^{j+1}r)} e^{-t\vec{\Delta}} \chi_{B(x, 2^{j+1}r)} \chi_{B(x,r)},$$

it is obvious that

$$\|\chi_{C_j(x,r)} e^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{p-q} \leq \|\chi_{B(x, 2^{j+1}r)} e^{-t\vec{\Delta}} \chi_{B(x, 2^{j+1}r)}\|_{p-q}. \quad (20)$$

Then (i) implies

$$\|\chi_{C_j(x,r)} e^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{p-q} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left[\max(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r}) \right]^\beta. \quad (21)$$

Using interpolation from (19) and (21), we deduce that

$$\|\chi_{C_j(x,r)} e^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{p-q} \leq \frac{C}{v(x,r)^{\frac{1}{p}-\frac{1}{q}}} \left[\max\left(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r}\right) \right]^\beta e^{-c \frac{\rho^2(C_j(x,r), B(x,r))}{t}},$$

and (ii) follows. \square

In the sequel we prove that the operators $d^* e^{-t\vec{\Delta}}$ and $d e^{-t\vec{\Delta}}$ satisfies L^p - L^2 off-diagonal estimates for all $p \in (p'_0, 2]$. We need the following lemma.

Lemma 4.2. *For any suitable ω and for every $x \in M$*

$$(i) \quad |d\omega(x)|_x \leq 2|\nabla\omega(x)|_x,$$

$$(ii) \quad |d^*\omega(x)|_x \leq \sqrt{N}|\nabla\omega(x)|_x.$$

Proof. As we did in the proof of **Lemma 3.2**, for every $x \in M$, we work in a synchronous frame to have an orthonormal basis $(\theta^i)_i$ of T_x^*M such that $\nabla\theta^i = 0$ at x . We recall that we have an inner product in each tensor product $T_x^*M \otimes T_x^*M$ satisfying

$$\langle \omega_1(x) \otimes \eta_1(x), \omega_2(x) \otimes \eta_2(x) \rangle_x = \langle \omega_1(x), \omega_2(x) \rangle_x \langle \eta_1(x), \eta_2(x) \rangle_x, \quad (22)$$

for all $\omega_1, \omega_2, \eta_1, \eta_2 \in \Lambda^1 T^*M$ and $x \in M$.

If $\omega(x) = f(x)\theta^i$ for a certain i , using (22), we have

$$\begin{aligned} |d\omega(x)|_x^2 &= |df(x) \wedge \theta^i|_x^2 = \left| \sum_{j=1}^n \partial_j f(x) \theta^j \wedge \theta^i \right|_x^2 \\ &= \sum_{j,k} \partial_j f(x) \partial_k f(x) \langle \theta^j \otimes \theta^i - \theta^i \otimes \theta^j, \theta^k \otimes \theta^i - \theta^i \otimes \theta^k \rangle_x \\ &= 2 \sum_j (\partial_j f(x))^2 - 2(\partial_i f(x))^2. \end{aligned}$$

Since $\sum_j (\partial_j f(x))^2 = |df(x)|_x^2$ at x , we obtain for $\omega(x) = f(x)\theta^i$

$$|d\omega(x)|_x^2 = 2(|df(x)|_x^2 - (\partial_i f(x))^2). \quad (23)$$

Now taking $\eta(x) = g(x)\theta^j$ for $j \neq i$, we have

$$\langle d\omega(x), d\eta(x) \rangle_x = \sum_{k,l} \partial_k f(x) \partial_l g(x) \langle \theta^k \otimes \theta^i - \theta^i \otimes \theta^k, \theta^l \otimes \theta^j - \theta^j \otimes \theta^l \rangle_x,$$

which, by (22), yields

$$\langle d\omega(x), d\eta(x) \rangle_x = -2\partial_j f(x)\partial_i g(x). \quad (24)$$

Thus, in the general case, writing $\omega(x) = \sum_i f_i(x)\theta^i = \sum_i \omega_i(x)$ and using (23) and (24), we obtain

$$\begin{aligned} |d\omega(x)|_x^2 &= \sum_i |d\omega_i(x)|_x^2 + \sum_{i \neq j} \langle d\omega_i(x), d\omega_j(x) \rangle_x \\ &= 2 \sum_i (|df_i(x)|_x^2 - (\partial_i f_i(x))^2) - 2 \sum_{i \neq j} \partial_j f_i(x) \partial_i f_j(x) \\ &= 2|\nabla\omega(x)|_x^2 - \sum_{i,j} (\partial_j f_i(x) + \partial_i f_j(x))^2 + 2 \sum_{i,j} (\partial_i f_j(x))^2 \\ &= 2|\nabla\omega(x)|_x^2 - \sum_{i,j} (\partial_j f_i(x) + \partial_i f_j(x))^2 + 2|\nabla\omega(x)|_x^2 \\ &\leq 4|\nabla\omega(x)|_x^2, \end{aligned}$$

which gives *i*). To prove *ii*), we notice that $d^*\omega(x) = -\sum_i \partial_i f_i(x)$ at x (see for instance [24] p.19). Hence using the Cauchy-Schwarz inequality and the previous calculations, we have

$$\begin{aligned} |d^*\omega(x)|_x^2 &\leq N \sum_i (\partial_i f_i(x))^2 \\ &= N \left(|\nabla\omega(x)|_x^2 - \frac{1}{4}|d\omega(x)|_x^2 - \frac{1}{4} \sum_{i \neq j} (\partial_j f_i(x) + \partial_i f_j(x))^2 \right) \\ &\leq N|\nabla\omega(x)|_x^2. \end{aligned}$$

□

We will need the following L^2 - L^2 off-diagonal estimate.

Proposition 4.3. *Assume that (S-C) is satisfied. Let E, F be two closed subsets of M . For any $\eta \in L^2(\Lambda^1 T^*M)$ with support in E we have*

$$\|\nabla e^{-t\vec{\Delta}} \eta\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-c \frac{\rho^2(E,F)}{t}} \|\eta\|_2.$$

Proof. As in the proof of **Proposition 3.6**, we set $\vec{\Delta}_\alpha = e^{\alpha\phi} \vec{\Delta} e^{-\alpha\phi}$ where $\alpha > 0$ is a constant and ϕ is a bounded Lipschitz function such that $|\nabla\phi(x)|_x \leq 1$ for almost every $x \in M$. Using the assumption (S-C), we obtain for $\omega \in \mathcal{D}(\vec{\Delta})$

$$\begin{aligned} ((\vec{\Delta}_\alpha + \alpha^2)\omega, \omega) &= \|H^{\frac{1}{2}}\omega\|_2^2 - \alpha^2 \int_M |\nabla\phi(x)|_x^2 |\omega(x)|_x^2 d\mu - (R_- \omega, \omega) + \alpha^2 \|\omega\|_x^2 \\ &\geq \|H^{\frac{1}{2}}\omega\|_2^2 - \alpha^2 \|\omega\|_2^2 - (R_- \omega, \omega) + \alpha^2 \|\omega\|_2^2 \\ &\geq (1 - \epsilon) \|H^{\frac{1}{2}}\omega\|_2^2 \\ &\geq (1 - \epsilon) \|\nabla\omega\|_2^2. \end{aligned}$$

We recall that from the proof of **Proposition 3.6**, one has for $\eta \in L^2(\Lambda^1 T^* M)$

$$\|e^{-t\vec{\Delta}_\alpha} \eta\|_2 \leq e^{t\alpha^2} \|\eta\|_2. \quad (25)$$

Lemma 4.4 below ensures that the operator $\vec{\Delta}_\alpha + 2\alpha^2$ is sectorial. As a consequence the semigroup $(e^{-z(\vec{\Delta}_\alpha + 2\alpha^2)})_{t \geq 0}$ is analytic on the sector $\Sigma = \{z \in \mathbb{C}, z \neq 0, |\arg(z)| \leq \frac{\pi}{2} - \text{Arctan}(\gamma)\}$ (where γ is the constant appearing in (29) below) and $\|e^{-z(\vec{\Delta}_\alpha + 2\alpha^2)}\|_{2,2} \leq 1$ for all $z \in \Sigma$ (see [23] Theorem 1.53, 1.54). A classical argument using the Cauchy formula implies that for all $t \geq 0$

$$\|(\vec{\Delta}_\alpha + 2\alpha^2)e^{-t(\vec{\Delta}_\alpha + 2\alpha^2)}\|_{2-2} \leq \frac{C}{t}, \quad (26)$$

where the constant C does not depend on α . We notice that for every $\omega \in \mathcal{D}(\vec{\Delta})$

$$((\vec{\Delta}_\alpha + 2\alpha^2)\omega, \omega) \geq ((\vec{\Delta}_\alpha + \alpha^2)\omega, \omega) \geq (1 - \epsilon) \|\nabla\omega\|_2^2. \quad (27)$$

Then setting $\omega = e^{-t(\vec{\Delta}_\alpha + 2\alpha^2)} \eta$ for $\eta \in L^2(\Lambda^1 T^* M)$ and $t \geq 0$, we deduce from (25), (26) and (27) that

$$\|\nabla e^{-t(\vec{\Delta}_\alpha + 2\alpha^2)} \eta\|_2 \leq \frac{C}{\sqrt{t}} \|e^{-t(\vec{\Delta}_\alpha + 2\alpha^2)} \eta\|_2 \leq \frac{C}{\sqrt{t}} \|\eta\|_2, \forall t > 0. \quad (28)$$

As we did in the proof of **Proposition 3.6** let E and F two closed subsets of M , $\eta \in L^2(\Lambda^1 T^* M)$ with support in E and $\phi_k(x) := \min(\rho(x, E), k)$ for $k \in \mathbb{N}$. Since $e^{\alpha\phi_k} \eta = \eta$, we have $e^{-t\vec{\Delta}} \eta = e^{-\alpha\phi_k} e^{-t\vec{\Delta}_\alpha} \eta$. Then we obtain

$$\nabla e^{-t\vec{\Delta}} \eta = -\alpha e^{-\alpha\phi_k} \nabla \phi_k \otimes e^{-t\vec{\Delta}_\alpha} \eta + e^{-\alpha\phi_k} \nabla e^{-t\vec{\Delta}_\alpha} \eta.$$

Since $|\nabla\phi_k(x)|_x \leq 1$ for almost every $x \in M$, we deduce from (25) and (28) that

$$\|\chi_F \nabla e^{-t\vec{\Delta}} \eta\|_2 \leq \alpha e^{-\alpha \min(\rho(E,F),k)} e^{t\alpha^2} \|\eta\|_2 + \frac{C}{\sqrt{t}} e^{-\alpha \min(\rho(E,F),k)} e^{2t\alpha^2} \|\eta\|_2.$$

Now letting k tends to infinity and setting $\alpha = \frac{\rho(E,F)}{4t}$, we finally obtain

$$\begin{aligned} \|\chi_F \nabla e^{-t\vec{\Delta}} \eta\|_2 &\leq \frac{C}{\sqrt{t}} \left(1 + \frac{\rho(E,F)}{4\sqrt{t}}\right) e^{-\frac{\rho^2(E,F)}{8t}} \|\eta\|_2 \\ &\leq \frac{C}{\sqrt{t}} e^{-c\frac{\rho^2(E,F)}{t}} \|\eta\|_2, \end{aligned}$$

which is the desired result. \square

In the following lemma, we study sectoriality. Then we need to work with complex valued 1-forms. This is achieved as usual by introducing the complex Hilbert spaces $L^2(\Lambda^1 T^* M) \oplus iL^2(\Lambda^1 T^* M)$ and $\mathcal{D}(\vec{a}) \oplus i\mathcal{D}(\vec{a})$.

Lemma 4.4. *Under the assumption (S-C), the operator $\vec{\Delta}_\alpha + 2\alpha^2$ is sectorial. That is there exists a constant $\gamma \geq 0$ such that for all $\omega \in \mathcal{D}(\vec{\Delta}_\alpha + 2\alpha^2)$*

$$|Im((\vec{\Delta}_\alpha + 2\alpha^2)\omega, \omega)| \leq \gamma Re((\vec{\Delta}_\alpha + 2\alpha^2)\omega, \omega) \quad (29)$$

Proof. We consider $\omega \in \mathcal{D}(\vec{a}) \oplus i\mathcal{D}(\vec{a})$. Since $|\nabla\phi(x)|_x \leq 1$ for almost every $x \in M$, we have

$$\begin{aligned} \vec{a}_\alpha(\omega, \omega) &= \vec{a}(\omega, \omega) + \alpha \int_M \langle \nabla\omega(x), \nabla\phi(x) \otimes \overline{\omega(x)} \rangle_x d\mu \\ &\quad - \alpha \int_M \langle \nabla\phi(x) \otimes \omega(x), \overline{\nabla\omega(x)} \rangle_x d\mu - \alpha^2 \int_M |\nabla\phi(x)|_x^2 |\omega(x)|_x^2 d\mu \\ &\geq \vec{a}(\omega, \omega) + 2i\alpha Im\left(\int_M \langle \nabla\phi(x) \otimes \omega(x), \overline{\nabla\omega(x)} \rangle_x d\mu\right) - \alpha^2 \|\omega\|_2^2. \end{aligned}$$

Therefore we deduce that

$$Re(\vec{a}_\alpha(\omega, \omega) + 2\alpha^2 \|\omega\|_2^2) \geq \vec{a}(\omega, \omega) \quad (30)$$

$$Re(\vec{a}_\alpha(\omega, \omega) + 2\alpha^2 \|\omega\|_2^2) \geq \alpha^2 \|\omega\|_2^2. \quad (31)$$

Furthermore, the Cauchy-Schwarz inequality and the assumption (S-C) yield

$$\begin{aligned}
|Im(\vec{a}_\alpha(\omega, \omega) + 2\alpha^2\|\omega\|_2^2)| &= \left| 2\alpha Im \left(\int_M \langle \nabla \phi(x) \otimes \omega(x), \overline{\nabla \omega(x)} \rangle_x d\mu \right) \right| \\
&\leq 2\alpha \int_M |\omega(x)|_x |\nabla \phi(x)|_x |\nabla \omega(x)|_x d\mu \\
&\leq 2\alpha \|\omega\|_2 \|\nabla \omega\|_2 \\
&\leq 2\alpha \|\omega\|_2 \|H^{\frac{1}{2}}\omega\|_2 \\
&\leq 2\alpha \sqrt{\frac{1}{1-\epsilon}} \|\omega\|_2 \vec{d}^{\frac{1}{2}}(\omega, \omega) \\
&\leq \frac{1}{1-\epsilon} \vec{d}(\omega, \omega) + \alpha^2 \|\omega\|_2^2.
\end{aligned}$$

Using (30) and (31), we deduce that there exists a constant C_ϵ such that

$$|Im(\vec{a}_\alpha(\omega, \omega) + 2\alpha^2\|\omega\|_2^2)| \leq C_\epsilon Re(\vec{a}_\alpha(\omega, \omega) + 2\alpha^2\|\omega\|_2^2),$$

which means that $\vec{\Delta}_\alpha + 2\alpha^2$ is sectorial. (see [23] Proposition 1.27) \square

An immediate consequence of **Lemma 4.2** and **Proposition 4.3** is the following result.

Corollary 4.5. *Assume that (S-C) is satisfied. Let E, F be two closed subsets of M . For any $\eta \in L^2(\Lambda^1 T^*M)$ with support in E*

$$(i) \quad \|de^{-t\vec{\Delta}}\eta\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-c\frac{\rho^2(E,F)}{t}} \|\eta\|_2,$$

$$(ii) \quad \|d^*e^{-t\vec{\Delta}}\eta\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-c\frac{\rho^2(E,F)}{t}} \|\eta\|_2.$$

We are now able to prove L^p - L^2 off-diagonal estimates for the operators $d^*e^{-t\vec{\Delta}}$ and $de^{-t\vec{\Delta}}$.

Theorem 4.6. *Suppose that (D), (G) and (S-C) are satisfied. Then for all $r, t > 0$, $x, y \in M$ and all $p \in (p'_0, 2]$*

$$\|\chi_{C_j(x,r)} de^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \frac{Ce^{-c\frac{4^j r^2}{t}}}{\sqrt{t} v(x,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max\left(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}\right) \right)^\beta 2^{j\beta}, \quad (32)$$

$$\|\chi_{C_j(x,r)} d^*e^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \frac{Ce^{-c\frac{4^j r^2}{t}}}{\sqrt{t} v(x,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max\left(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}\right) \right)^\beta 2^{j\beta}, \quad (33)$$

where $C_j(x, r) = B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ and $\beta \geq 0$ depends on p .

Proof. We only prove (32) since (33) can be obtained in the same manner. By **Corollary 4.5**, we have for all $x, z \in M$ and $r, t \geq 0$

$$\|\chi_{B(x,r)} d e^{-t\vec{\Delta}} \chi_{B(z,r)}\|_{2-2} \leq \frac{C}{\sqrt{t}} e^{-c \frac{\rho^2(B(x,r), B(z,r))}{t}}.$$

In addition by **Theorem 4.1**, we have for all $y, z \in M$, $r, t \geq 0$ and $p \in (p'_0, 2]$

$$\|\chi_{B(z,r)} e^{-t\vec{\Delta}} \chi_{B(y,r)}\|_{p-2} \leq \frac{C}{v(z,r)^{\frac{1}{p}-\frac{1}{2}}} e^{-c \frac{\rho^2(B(y,r), B(z,r))}{t}}.$$

Then writing $d e^{-t\vec{\Delta}} = d e^{-\frac{t}{2}\vec{\Delta}} e^{-\frac{t}{2}\vec{\Delta}}$ and using a composition argument, we obtain

$$\|\chi_{B(x,r)} d e^{-t\vec{\Delta}} \chi_{B(y,r)}\|_{p-2} \leq \frac{C}{\sqrt{t} v(y,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max\left(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}\right) \right)^\beta e^{-c \frac{\rho^2(B(x,r), B(y,r))}{t}}. \quad (34)$$

For more details on the composition argument see [2] Theorem 3.5.

Writing $\chi_{C_j(x,r)} d e^{-t\vec{\Delta}} \chi_{B(x,r)} = \chi_{C_j(x,r)} \chi_{B(x,2^{j+1}r)} d e^{-t\vec{\Delta}} \chi_{B(x,2^{j+1}r)} \chi_{B(x,r)}$, we notice that

$$\|\chi_{C_j(x,r)} d e^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{p-2} \leq \|\chi_{B(x,2^{j+1}r)} d e^{-t\vec{\Delta}} \chi_{B(x,2^{j+1}r)}\|_{p-2}$$

Then (34) yields

$$\begin{aligned} \|\chi_{C_j(x,r)} d e^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{p-2} &\leq \frac{C}{\sqrt{t} v(y,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max\left(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r}\right) \right)^\beta \\ &\leq \frac{C 2^{j\beta}}{\sqrt{t} v(y,r)^{\frac{1}{p}-\frac{1}{2}}} \left(\max\left(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}\right) \right)^\beta. \end{aligned}$$

Using **Corollary 4.5**, we have

$$\|\chi_{C_j(x,r)} d e^{-t\vec{\Delta}} \chi_{B(x,r)}\|_{2-2} \leq \frac{C}{\sqrt{t}} e^{-c \frac{4^j r^2}{t}}. \quad (35)$$

Therefore applying the Riesz-Thorin interpolation theorem from (34) and (35), we deduce the result. \square

A key result to prove the boundedness of the Riesz transforms $d^*(\vec{\Delta})^{-\frac{1}{2}}$ and $d(\vec{\Delta})^{-\frac{1}{2}}$ is a result in [8] which we state as it is formulated in [3], Theorem 2.1.

Theorem 4.7. *Let $p \in (1, 2]$. Suppose that T is a sublinear operator of strong type $(2, 2)$, and let $(A_r)_{r>0}$ be a family of linear operators acting on L^2 . Assume that for $j \geq 2$ and every ball $B = B(x, r)$*

$$\left(\frac{1}{v(x, 2^{j+1}r)} \int_{C_j(x, r)} |T(I - A_r)f|^2 \right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{v(x, r)} \int_B |f|^p \right)^{\frac{1}{p}}, \quad (36)$$

and for $j \geq 1$

$$\left(\frac{1}{v(x, 2^{j+1}r)} \int_{C_j(x, r)} |A_r f|^2 \right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{v(x, r)} \int_B |f|^p \right)^{\frac{1}{p}}, \quad (37)$$

for all f supported in B . If $\Sigma := \sum_j g(j) 2^{Dj} < \infty$, then T is of weak type (p, p) , with a bound depending only on the strong type $(2, 2)$ bound of T , p and Σ .

Finally we prove **Theorem 1.1**.

Proof of Theorem 1.1. We argue as in [2] Theorem 3.6. We set $T = d^*(\vec{\Delta})^{-\frac{1}{2}}$ and consider the operators $A_r = I - (I - e^{-r^2 \vec{\Delta}})^m$ for some sufficiently large integer m . The estimate (37) can be obtained using the estimate

$$\|\chi_{C_j(x, r)} e^{-t \vec{\Delta}} \chi_{B(x, r)}\|_{p-q} \leq \frac{C e^{-c \frac{4^j r^2}{t}}}{v(x, r)^{\frac{1}{p}-\frac{1}{q}}} \left(\max\left(\frac{2^{j+1}r}{\sqrt{t}}, \frac{\sqrt{t}}{2^{j+1}r}\right) \right)^\beta,$$

which we proved in **Theorem 4.1** (see [2] Theorem 3.6).

The estimate (36) can be obtained using the estimate

$$\|\chi_{C_j(x, r)} d^* e^{-t \vec{\Delta}} \chi_{B(x, r)}\|_{p-2} \leq \frac{C e^{-c \frac{4^j r^2}{t}}}{\sqrt{t} v(x, r)^{\frac{1}{p}-\frac{1}{q}}} \left(\max\left(\frac{r}{\sqrt{t}}, \frac{\sqrt{t}}{r}\right) \right)^\beta 2^{j\beta},$$

which we proved in **Theorem 4.6** (see [2] Theorem 3.6).

The proof is the same for $T = d(\vec{\Delta})^{-\frac{1}{2}}$. □

5 Sub-criticality and proof of Theorem 1.3

The assumption (S-C) can be understood as a "smallness" condition on the negative part R_- of the Ricci curvature. But since R_- is a geometric component of the

manifold M , it would be interesting to have analytic or geometric conditions which lead to this assumption. This is the purpose of this section.

We recall that Devyver [17] studied the boundedness of the Riesz transform $d(\Delta)^{-\frac{1}{2}}$ from $L^p(M)$ to $L^p(\Lambda^1 T^*M)$ where M is a complete non-compact Riemannian manifold satisfying a global Sobolev type inequality

$$\|f\|_{\frac{2N}{N-2}} \leq C\|df\|_2, \forall f \in \mathcal{C}_0^\infty(M).$$

Assuming $R_- \in L^{\frac{N}{2}}$, he proved that R_- satisfies the assumption (S-C) if and only if the space

$$\text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) := \{\omega \in \mathcal{D}(\vec{\mathfrak{h}}) : \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M), (\omega, \vec{\Delta}\eta) = 0\}$$

is trivial. Here \mathfrak{h} denotes the sesquilinear form defined for all $\omega, \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$ by

$$\vec{\mathfrak{h}}(\omega, \eta) = \int_M \langle \nabla \omega(x), \nabla \eta(x) \rangle_x d\mu + \int_M \langle R_+(x)\omega(x), \eta(x) \rangle_x d\mu,$$

$$\text{and } \mathcal{D}(\vec{\mathfrak{h}}) = \overline{\mathcal{C}_0^\infty(\Lambda^1 T^*M)}^{\|\cdot\|_{\vec{\mathfrak{h}}}},$$

where $\|\omega\|_{\vec{\mathfrak{h}}} = \sqrt{\vec{\mathfrak{h}}(\omega, \omega) + \|\omega\|_2^2}$. We recall that H denotes its associated operator, that is, $H = \nabla^* \nabla + R_+$.

Assaad and Ouhabaz introduced in [2] the following quantities

$$\alpha_1 = \int_0^1 \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{dt}{\sqrt{t}}, \quad \alpha_2 = \int_1^\infty \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}},$$

for some $r_1, r_2 > 2$. We set $\|R_-^{\frac{1}{2}}\|_{vol} := \alpha_1 + \alpha_2$. We are interested in the finiteness of this norm. It is clear that if the volume is polynomial, that is, $cr^N \leq v(x, r) \leq Cr^N$, then $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$ if and only if $R_- \in L^{\frac{N}{2}-\eta} \cap L^{\frac{N}{2}+\eta}$ for some $\eta > 0$. The latter condition is usually assumed to study the boundedness of Riesz transforms of Schrödinger operators on L^p for $p > 2$.

We state the main result of this section.

Theorem 5.1. *Assume that the manifold M satisfies (D), (G) and $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$. Then R_- satisfies (S-C) if and only if $\text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\}$.*

We can observe that this result is similar to the one of Devyver. However, we do not assume any global Sobolev inequality. In this context, with the additional assumption that the balls of great radius has polynomial volume growth, Definition 2.2.2 in [17] allows $R_- \in L^{\frac{N}{2}}$; whereas in **Theorem 5.1**, one needs $R_- \in L^{\frac{N}{2}-\eta} \cap L^{\frac{N}{2}+\eta}$ for some $\eta > 0$ with the same condition on the volume.

Assuming **Theorem 5.1**, we are now able to prove **Theorem 1.3**.

Proof of Theorem 1.3. According to the commutation formula $\vec{\Delta}d = d\Delta$, we see that the adjoint operator of $d^*(\vec{\Delta})^{-\frac{1}{2}}$ is exactly $d(\Delta)^{-\frac{1}{2}}$. Then **Theorem 1.3** is an immediate consequence of **Theorem 5.1** and **Theorem 1.1**. \square

Let us make a comment on the space $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta})$. We consider $\omega \in \mathcal{D}(\vec{\mathfrak{h}})$. Since $\vec{\Delta}$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\Lambda^1 T^*M)$ (see [26] Section 2), the condition

$$(\omega, \vec{\Delta}\eta) = 0, \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$$

implies

$$(\omega, \vec{\Delta}\eta) = 0, \forall \eta \in \mathcal{D}(\vec{\Delta}).$$

Then $\omega \in \mathcal{D}(\vec{\Delta})$ and $\vec{\Delta}\omega = 0$. Therefore $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta})$ is the space of harmonic L^2 forms.

The following proposition proves the first part of **Theorem 5.1**.

Proposition 5.2. *Assume that M satisfies (D), (G) and that R_- satisfies (S-C). Then $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\}$.*

Proof. Any ω in $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta})$ satisfies for all $\eta \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$, $(\vec{\Delta}\omega, \eta) = 0$, hence, by a density argument $(\vec{\Delta}\omega, \omega) = 0$. If R_- satisfies (S-C), we have $(H\omega, \omega) \leq \frac{1}{1-\epsilon}(\vec{\Delta}\omega, \omega) = 0$, which yields $\omega \in Ker(H^{\frac{1}{2}})$. According to **Lemma 5.3** below, we deduce that $\omega = 0$. Thus $Ker_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\}$. \square

The following result is well-known but we have decided to give its proof for the sake of completeness.

Lemma 5.3. *Assume that (D) and (G) are satisfied. Then $Ker(H) = \{0\}$.*

Proof. We consider $\omega \in \text{Ker}(H)$, that is $\omega \in \mathcal{D}(H)$ and $H\omega = 0$. We then have for all $t \geq 0$

$$e^{-tH}\omega = \omega. \quad (38)$$

Noticing that we have the domination $|e^{-tH}\omega| \leq e^{-t\Delta}|\omega|$ and using (38) and (G), we obtain for all $x \in M$ and $t \geq 0$

$$|\omega(x)|_x \leq \frac{C}{v(x, \sqrt{t})} \int_M \exp(-c \frac{\rho^2(x, y)}{t}) |\omega(y)|_y d\mu.$$

The Hölder inequality yields

$$|\omega(x)|_x \leq \frac{C}{v(x, \sqrt{t})} \left(\int_M \exp(-2c \frac{\rho^2(x, y)}{t}) d\mu \right)^{\frac{1}{2}} \|\omega\|_2. \quad (39)$$

Using (12) in (39) leads to

$$|\omega(x)|_x \leq \frac{C}{\sqrt{v(x, \sqrt{t})}} \|\omega\|_2. \quad (40)$$

Since the manifold M is connected, complete, non-compact and satisfies the doubling volume property (D), it follows from [19] p.412 that there exists a constant $D' > 0$ such that for all $x \in M$ and $0 < r \leq R$

$$\frac{v(x, R)}{v(x, r)} \geq c \left(\frac{R}{r} \right)^{D'}. \quad (41)$$

We obtain from (40) and (41) that for all $t \geq 1$

$$|\omega(x)|_x \leq \frac{C}{t^{\frac{D'}{4}} \sqrt{v(x, 1)}} \|\omega\|_2.$$

Letting t tend to infinity, we deduce that for all $x \in M$, $|\omega(x)|_x = 0$ and then that $\text{Ker}(H) = \{0\}$. □

Note that the assumption $\|R^{\frac{1}{2}}\|_{vol} < \infty$ is not necessary in the proof of **Proposition 5.2** but will be used to prove the converse of **Theorem 5.1**.

Before giving the other half of the proof of **Theorem 5.1**, we need the following two results.

Lemma 5.4. *Assume that (D) and (G) are satisfied. Then there exists a constant $C \geq 0$ such that*

$$\|R_-^{\frac{1}{2}} H^{-\frac{1}{2}}\|_{2-2} \leq C \|R_-^{\frac{1}{2}}\|_{vol}$$

and

$$\|H^{-\frac{1}{2}} R_-^{\frac{1}{2}}\|_{2-2} \leq C \|R_-^{\frac{1}{2}}\|_{vol}.$$

Proof. Writing $H^{-\frac{1}{2}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-tH} \frac{dt}{\sqrt{t}}$ and using the Hölder inequality, we obtain

$$\begin{aligned} & \|R_-^{\frac{1}{2}} H^{-\frac{1}{2}}\|_{2-2} \\ & \leq C \int_0^1 \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_1}}} v(\cdot, \sqrt{t})^{\frac{1}{r_1}} e^{-tH} \right\|_{2-2} \frac{dt}{\sqrt{t}} + C \int_1^\infty \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_2}}} v(\cdot, \sqrt{t})^{\frac{1}{r_2}} e^{-tH} \right\|_{2-2} \frac{dt}{\sqrt{t}} \\ & \leq C \int_0^1 \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \left\| v(\cdot, \sqrt{t})^{\frac{1}{r_1}} e^{-tH} \right\|_{2-\frac{2r_1}{r_1-2}} \frac{dt}{\sqrt{t}} \\ & \quad + C \int_1^\infty \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \left\| v(\cdot, \sqrt{t})^{\frac{1}{r_2}} e^{-tH} \right\|_{2-\frac{2r_2}{r_2-2}} \frac{dt}{\sqrt{t}} \end{aligned}$$

and similarly

$$\begin{aligned} & \|H^{-\frac{1}{2}} R_-^{\frac{1}{2}}\|_{2-2} \\ & \leq C \int_0^1 \left\| e^{-tH} v(\cdot, \sqrt{t})^{\frac{1}{r_1}} \right\|_{\frac{2r_1}{r_1+2}-2} \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{dt}{\sqrt{t}} \\ & \quad + C \int_1^\infty \left\| e^{-tH} v(\cdot, \sqrt{t})^{\frac{1}{r_2}} \right\|_{\frac{2r_2}{r_2+2}-2} \left\| \frac{R_-^{\frac{1}{2}}}{v(\cdot, \sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{dt}{\sqrt{t}}. \end{aligned}$$

The assumptions (D) and (G) allow us to use Proposition 2.9 in [2] for Δ . Then noticing we have the domination $|e^{-tH}\omega| \leq e^{-t\Delta}|\omega|$, for all $\omega \in \mathcal{C}_0^\infty(\Lambda^1 T^*M)$ leads to the following estimates

$$\|v(\cdot, \sqrt{t})^{\frac{1}{p}-\frac{1}{q}} e^{-tH}\|_{p-q} \leq C, \quad \forall 1 < p \leq q < \infty,$$

where C is a non-negative constant depending on p, q , (D) and (G). By duality

$$\|e^{-tH} v(\cdot, \sqrt{t})^{\frac{1}{p}-\frac{1}{q}}\|_{p-q} \leq C, \quad \forall 1 < p \leq q < \infty.$$

Since for $i = 1, 2$ we have $\frac{1}{r_i} = \frac{1}{2} - \frac{r_i-2}{2r_i}$ and $\frac{1}{r_i} = \frac{r_i+2}{2r_i} - \frac{1}{2}$, we obtain the desired result. \square

As a consequence

Corollary 5.5. *The L^2 -adjoint of the operator $R_-^{\frac{1}{2}}H^{-\frac{1}{2}}$ is $H^{-\frac{1}{2}}R_-^{\frac{1}{2}}$.*

We now follow the ideas of Devyver to prove **Theorem 5.1**. Even if the two lemmas below are known, we give their proofs for the sake of completeness. The following lemma is similar to Lemma 1 in [17].

Lemma 5.6. *Let Λ denote the self-adjoint operator $H^{-\frac{1}{2}}R_-H^{-\frac{1}{2}} = (R_-^{\frac{1}{2}}H^{-\frac{1}{2}})^*(R_-^{\frac{1}{2}}H^{-\frac{1}{2}})$ acting on $L^2(\Lambda^1T^*M)$. Assume that (D) and (G) are satisfied. Then the operator $H^{\frac{1}{2}}$ is an isomorphism from $\text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta})$ to $\text{Ker}_{L^2}(I - \Lambda)$.*

Proof. We consider $\omega \in \text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta})$, that is, $\omega \in \mathcal{D}(\vec{\mathfrak{h}})$ such that for all $\eta \in \mathcal{C}_0^\infty(\Lambda^1T^*M)$

$$(\omega, \vec{\Delta}\eta) = 0.$$

Let $\eta \in \mathcal{C}_0^\infty(\Lambda^1T^*M)$. We write $\vec{\Delta}\eta = H^{\frac{1}{2}}(I - \Lambda)H^{\frac{1}{2}}\eta$. Since $\mathcal{D}(\vec{\mathfrak{h}}) = \mathcal{D}(H^{\frac{1}{2}})$, we may write

$$\omega \in \text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) \iff \forall \eta \in \mathcal{C}_0^\infty(\Lambda^1T^*M), (H^{\frac{1}{2}}\omega, (I - \Lambda)H^{\frac{1}{2}}\eta) = 0.$$

We claim that $H^{\frac{1}{2}}(\mathcal{C}_0^\infty(\Lambda^1T^*M))$ is dense in $L^2(\Lambda^1T^*M)$. Assuming the claim, we obtain

$$\omega \in \text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) \iff (H^{\frac{1}{2}}\omega, (I - \Lambda)\eta) = 0, \forall \eta \in L^2(\Lambda^1T^*M).$$

Noticing that $I - \Lambda$ is self-adjoint on $L^2(\Lambda^1T^*M)$, we deduce that

$$\omega \in \text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) \iff H^{\frac{1}{2}}\omega \in \text{Ker}_{L^2}(I - \Lambda).$$

Now we prove the claim. We consider $u = H^{\frac{1}{2}}v \in \text{Im}(H^{\frac{1}{2}})$ satisfying

$$(u, H^{\frac{1}{2}}w) = 0, \forall w \in \mathcal{C}_0^\infty(\Lambda^1T^*M).$$

Then for all $w \in \mathcal{C}_0^\infty(\Lambda^1T^*M)$, we have $\vec{\mathfrak{h}}(v, w) = 0$. Therefore $v \in \mathcal{D}(H)$ and $Hv = 0$, that is $v \in \text{Ker}(H)$. Since $\text{Ker}(H) = \{0\}$ (see **Lemma 5.3** below), we obtain $v = 0$ in $\mathcal{D}(H)$ and then $u = 0$ in $\text{Im}(H^{\frac{1}{2}})$. This shows that $H^{\frac{1}{2}}(\mathcal{C}_0^\infty(\Lambda^1T^*M))$ is dense in $\text{Im}(H^{\frac{1}{2}})$. Furthermore, $\text{Im}(H^{\frac{1}{2}})$ is dense in $L^2(\Lambda^1T^*M)$ because $H^{\frac{1}{2}}$ is self-adjoint and $\text{Ker}(H^{\frac{1}{2}}) = \{0\}$. Hence we deduce that $H^{\frac{1}{2}}(\mathcal{C}_0^\infty(\Lambda^1T^*M))$ is dense in $L^2(\Lambda^1T^*M)$. \square

The following lemma is similar to Proposition 1.4 and Theorem 1.5 in [11].

Lemma 5.7. *Assume that the manifold M satisfies (D), (G) and $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$. Then Λ is a compact operator on $L^2(\Lambda^1 T^* M)$.*

Proof. It follows from the same proof as in **Lemma 5.4**, applied to $\chi_{B(x,r)^C} R_-^{\frac{1}{2}}$ rather than $R_-^{\frac{1}{2}}$, that we have for all $x \in M$ and $r \geq 0$

$$\|\chi_{B(x,r)^C} R_-^{\frac{1}{2}} H^{-\frac{1}{2}}\|_{2-2} \leq C \|\chi_{B(x,r)^C} R_-^{\frac{1}{2}}\|_{vol},$$

where $B(x,r)^C$ denotes $M \setminus B(x,r)$. In addition the dominated convergence theorem applied twice ensures that for all $x \in M$

$$\lim_{r \rightarrow +\infty} \|\chi_{B(x,r)^C} R_-^{\frac{1}{2}}\|_{vol} = 0.$$

Therefore we deduce that

$$\lim_{r \rightarrow +\infty} \chi_{B(x,r)} R_-^{\frac{1}{2}} H^{-\frac{1}{2}} = R_-^{\frac{1}{2}} H^{-\frac{1}{2}},$$

where the limit is the operator limit in $\mathcal{L}(L^2(\Lambda^1 T^* M))$.

We recall that the operator limit in the uniform sense of compact operators is compact. Then to prove the lemma, it suffices to show that the operator $\chi_{B(x,r)} R_-^{\frac{1}{2}} H^{-\frac{1}{2}}$ is compact on $L^2(\Lambda^1 T^* M)$ for all $x \in M$ and $r \geq 0$. Since R_- is continuous on M , $R_- \in L_{loc}^\infty(M)$ and then there exists $\phi \in \mathcal{C}_0^\infty(M)$ such that $\phi = 1$ on $B(x,r)$, $\phi \leq 1$ on $B(x,r)^C$ and

$$\|\chi_{B(x,r)} R_-^{\frac{1}{2}} H^{-\frac{1}{2}} \omega\|_2 \leq C \|\phi H^{-\frac{1}{2}} \omega\|_2, \forall \omega \in L^2(\Lambda^1 T^* M),$$

where $C = \max_{x \in \text{supp}(\phi)} \|R_-^{\frac{1}{2}}(x)\|$. It suffices then to prove that the operator $\phi H^{-\frac{1}{2}}$ is compact on $L^2(\Lambda^1 T^* M)$. We recall that we have a compact embedding between the Sobolev space $W^{1,2}(\Lambda^1 T^* K)$ and the space $L^2(\Lambda^1 T^* M)$ for all compact subsets K of M (see [24] p.24, 27, 34). Since ϕ has compact support and $\text{Im}(H^{-\frac{1}{2}}) = \mathcal{D}(\vec{\mathfrak{h}}) \subseteq W^{1,2}(\Lambda^1 T^* M)$, we deduce that the operator $\phi H^{-\frac{1}{2}}$ is compact on $L^2(\Lambda^1 T^* M)$.

We conclude that $\Lambda = (R_-^{\frac{1}{2}} H^{-\frac{1}{2}})^* (R_-^{\frac{1}{2}} H^{-\frac{1}{2}})$ is compact on $L^2(\Lambda^1 T^* M)$. \square

We are now able to end the proof of **Theorem 5.1**.

Proof of Theorem 5.1. First we notice that $\vec{\Delta}$ being positive on $L^2(\Lambda^1 T^* M)$, we have for all $\omega \in \mathcal{D}(\vec{\mathfrak{h}})$

$$(R_- \omega, \omega) \leq (H \omega, \omega).$$

Then for all $\omega \in L^2(\Lambda^1 T^* M)$

$$(\Lambda \omega, \omega) = (H^{-\frac{1}{2}} R_- H^{-\frac{1}{2}} \omega, \omega) \leq \|\omega\|_2^2.$$

Hence

$$\|\Lambda\|_{2-2} \leq 1. \quad (42)$$

According to the self-adjointness and the positivity of Λ , we have

$$\|\Lambda\|_{2-2} = \max\{\lambda; \lambda \text{ eigenvalue of } \Lambda\}. \quad (43)$$

Furthermore, **Lemma 5.7** and the Fredholm alternative imply

$$1 \text{ is an eigenvalue of } \Lambda \iff \text{Ker}_{L^2}(I - \Lambda) \neq \{0\}, \quad (44)$$

whereas **Lemma 5.6** ensures that

$$\text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\} \iff \text{Ker}_{L^2}(I - \Lambda) = \{0\}. \quad (45)$$

Therefore we deduce from (42), (43), (44) and (45) that

$$\text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\} \iff \|\Lambda\|_{2-2} < 1.$$

Since Λ is self-adjoint on $L^2(\Lambda^1 T^* M)$, note that

$$R_- \text{ is } \epsilon\text{-sub-critical} \iff \exists 0 \leq \epsilon < 1, \|\Lambda\|_{2-2} \leq \epsilon.$$

The result follows. □

The following results aim at removing the assumption $\text{Ker}_{\mathcal{D}(\vec{\mathfrak{h}})}(\vec{\Delta}) = \{0\}$. However we need to strengthen the assumption on $\|R_-^{\frac{1}{2}}\|_{vol}$. We start with a proposition.

Proposition 5.8. *Assume that the manifold M satisfies (D), (G) and $\|R_-^{\frac{1}{2}}\|_{vol} < \infty$. Then there exists a non-negative constant C depending on the constants appearing in (D) and (G) such that for any $\omega \in \mathcal{D}(\vec{\mathfrak{h}})$*

$$(R_- \omega, \omega) \leq C \|R_-^{\frac{1}{2}}\|_{vol}^2 \vec{\mathfrak{h}}(\omega, \omega) = C \|R_-^{\frac{1}{2}}\|_{vol}^2 (H \omega, \omega).$$

Proof. We have

$$(R_-\omega, \omega) = \|R_-^{\frac{1}{2}}\omega\|_2^2 = \|R_-^{\frac{1}{2}}H^{-\frac{1}{2}}H^{\frac{1}{2}}\omega\|_2^2 \leq \|R_-^{\frac{1}{2}}H^{-\frac{1}{2}}\|_{2-2}^2 \|H^{\frac{1}{2}}\omega\|_2^2.$$

Using **Lemma 5.4**, we obtain the desired result. \square

An immediate consequence of **Proposition 5.8** is the following.

Proposition 5.9. *Suppose that the assumptions (D) and (G) are satisfied and that $\|R_-^{\frac{1}{2}}\|_{vol}$ is small enough. Then R_- satisfies (S-C).*

In the particular case of polynomial volume growth, we then ask $\|R_-\|_{\frac{N}{2}-\eta}$ and $\|R_-\|_{\frac{N}{2}+\eta}$ to be small enough for some $\eta > 0$ to have R_- satisfying (S-C). Note that if M satisfies a global Sobolev inequality, it is easy to prove that R_- satisfies (S-C) if $\|R_-\|_{\frac{N}{2}}$ is small enough (without any assumption on the volume growth).

Note also that we recover $Ker_{\mathcal{D}(\vec{h})}(\vec{\Delta}) = \{0\}$ with the assumptions of **Proposition 5.9** but we did not need to assume it to prove subcriticality.

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